

# GENERALIZED ACTION PRINCIPLE and GEOMETRIC APPROACH for SUPERSTRINGS and SUPER- $P$ -BRANES

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Most of results described in this talk were obtained in collaboration with Dmitriy P. Sorokin and our teacher Dmitriy V. Volkov who, to our great sorrow, untimely leaved us this January.

In this short printed version of my talk I describe the generalized action for super- $p$ -branes [1] which can be used to construct the doubly supersymmetric generalization [2] of the geometric approach [3], and discuss one of the direction of its possible application [4].

1. The generalized action for super- $p$ - brane [1]

$$S_{D,p} = \int_{\mathcal{M}^{p+1}} \mathcal{L}_{p+1} = \int_{\mathcal{M}^{p+1}} \left( -\frac{(-1)^p}{p!} (E^a e^{a_1} \dots e^{a_p} - \frac{p}{(p+1)} e^a e^{a_1} \dots e^{a_p}) \varepsilon_{aa_1 \dots a_p} + \mathcal{L}_{p+1}^{WZ} \right) \quad (1)$$

is the integral of  $(p+1)$ - form  $\mathcal{L}_{p+1}$  over arbitrary  $(p+1)$  - dimensional bosonic surface  $\mathcal{M}^{p+1} = \{(\xi^m, \eta^{\mu q}) : \eta^{\mu q} = \eta^{\mu q}(\xi)\}$  in the world volume superspace  $\Sigma^{(p+1|n)} = \{(\xi^m, \eta^{\mu q})\}$  of the super-  $p$ - brane ( $m = 0, 1, \dots, p$ ). Lagrangian two- form (1) contains the Wess- Zumino term  $\mathcal{L}_{p+1}^{WZ}$  ( $d\mathcal{L}_{p+1}^{WZ} = -i\Pi^{\underline{m}_{p+1}} \dots \Pi^{\underline{m}_1} d\Theta_{\underline{m}_1 \dots \underline{m}_p} d\Theta$ ) and is constructed from some of the basic one forms of target superspace

$$\Pi^{\underline{m}} = dX^{\underline{m}} - id\Theta \Gamma_{\underline{m}} \Theta, \quad d\Theta^{\underline{\mu}} \quad \underline{m} = 0, \dots, (D-1) \quad (2)$$

$$E^{\underline{a}} = (E^a, E^i) = \Pi^{\underline{m}} u_{\underline{m}}^{\underline{a}} = (\Pi^{\underline{m}} u_{\underline{m}}^a, \Pi^{\underline{m}} u_{\underline{m}}^i) \quad \underline{a} = 0, \dots, (D-1); \quad a = 0, \dots, p \quad (3)$$

$$E^{\underline{\alpha}} = (E^{\alpha q}, E^{\alpha \dot{q}}) = d\Theta^{\underline{\mu}} v_{\underline{\mu}}^{\underline{\alpha}} = (d\Theta^{\underline{\mu}} v_{\underline{\mu}}^{\alpha q}, d\Theta^{\underline{\mu}} v_{\underline{\mu}}^{\alpha \dot{q}}), \quad (4)$$

and world volume superspace  $e^A = (e^a, e^{\alpha q}) = d\xi^m e_m^A + d\eta^{\mu q} e_{\mu q}^A$  using the external product of the forms only. Supervielbein of flat target superspace  $E^{\underline{A}} = (E^{\underline{a}}, E^{\underline{\alpha}})$  (3), (4) differs from the standard one (2) by Lorentz rotation, which vector and spinor representations are given by the matrices  $u_{\underline{m}}^{\underline{a}}$  and  $v_{\underline{\mu}}^{\underline{\alpha}}$

$$||u_{\underline{m}}^{\underline{a}}|| = ||(u_{\underline{m}}^a, u_{\underline{m}}^i)|| \in SO(1, D-1) \quad \Leftrightarrow \quad u_{\underline{m}}^{\underline{a}} \eta^{\underline{mn}} u_{\underline{n}}^{\underline{b}} = \eta^{\underline{ab}}$$

$$||v_{\underline{\mu}}^{\underline{\alpha}}|| = ||(v_{\underline{\mu}}^{\alpha q}, v_{\underline{\mu}}^{\alpha \dot{q}})|| \in Spin(1, D-1)$$

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Hence,  $u$  and  $v$  matrices (vector and spinor Lorentz harmonics, see [5,6,2] and refs. in [2]) are related by the conditions of the conservation of the  $D$ -dimensional  $\gamma$ -matrices

$$u_{\underline{m}}^{\underline{a}} \Gamma_{\underline{\mu}\underline{\nu}}^{\underline{m}} = v_{\underline{\mu}}^{\underline{\alpha}} \Gamma_{\underline{\alpha}\underline{\beta}}^{\underline{a}} v_{\underline{\nu}}^{\underline{\beta}}, \quad u_{\underline{m}}^{\underline{a}} \Gamma_{\underline{a}}^{\underline{\alpha}\underline{\beta}} = v_{\underline{\mu}}^{\underline{\alpha}} \Gamma_{\underline{m}}^{\underline{\mu}\underline{\nu}} v_{\underline{\nu}}^{\underline{\beta}},$$

Their differentials  $du_{\underline{m}}^{\underline{a}} = u_{\underline{m}}^{\underline{b}} \Omega_{\underline{b}}^{\underline{a}}(d)$ ,  $dv_{\underline{\mu}}^{\underline{\alpha}} \propto 1/4 v_{\underline{\mu}}^{\underline{\beta}} (\Gamma_{\underline{ab}})_{\underline{\beta}}^{\underline{\alpha}} \Omega^{\underline{ab}}(d)$ , are expressed in terms of the  $so(1, D-1)$  valued Cartan 1-form

$$\Omega^{\underline{ab}} = -\Omega^{\underline{ba}} = \begin{pmatrix} \Omega^{\underline{ab}} & \Omega^{\underline{aj}} \\ -\Omega^{\underline{bi}} & \Omega^{\underline{ij}} \end{pmatrix} = u_{\underline{m}}^{\underline{a}} du^{\underline{bm}} \propto dv_{\underline{\mu}}^{\underline{\alpha}} (\Gamma^{\underline{ab}})_{\underline{\alpha}}^{\underline{\beta}} v_{\underline{\beta}}^{\underline{\mu}} \quad (5)$$

In the functional (1) all the variables shall be considered as world volume superfields  $X^{\underline{m}} = X^{\underline{m}}(\xi, \eta)$ ,  $\Theta^{\underline{\mu}} = \Theta^{\underline{\mu}}(\xi, \eta)$ ,  $u_{\underline{m}}^{\underline{a}} = u_{\underline{m}}^{\underline{a}}(\xi, \eta)$ ,  $e_{\underline{M}}^{\underline{A}} = e_{\underline{M}}^{\underline{A}}(\xi, \eta)$ , ... but taken on the surface  $\mathcal{M}^{(p+1)}$ :  $\eta = \eta(\xi)$ ,  $X^{\underline{m}} = X^{\underline{m}}(\xi, \eta(\xi))$ , ...  $e_{\underline{M}}^{\underline{A}} = e_{\underline{M}}^{\underline{A}}(\xi, \eta(\xi))$ , ...

The generalized action concept consists in the requirement that the variation of the functional (1) should vanishes for arbitrary variations of the (super)fields involved as well as for arbitrary variations of the surface  $\mathcal{M}^{(p+1)}$ . For the Lagrangian form under consideration it can be proved (see [1] and Refs. therein) that the variation with respect to the surface  $\mathcal{M}^{(p+1)}$  (i.e.  $\delta S/d\eta(\xi) = 0$ ) does not lead to new equations of motion. However, the arbitrariness of surface  $\mathcal{M}^{(p+1)}$  gives the possibility to consider another equations of motion ( $\delta S/\delta\Theta = 0$ , etc. ) as superfield equations, i.e. as equations for the forms and superfields defined on the whole world volume superspace  $\Sigma^{(p+1|n)}$ .

The detailed consideration of the properties of the generalized action (1) can be found in Refs. [1].

**2.** A pure bosonic limit [6]  $S_{D,p}^0$  of the generalized action (1) is provided by the substitution  $\mathcal{M}^{(p+1)} \rightarrow \mathcal{M}^{(p+1)} = \{(\xi^m, \eta^{\mu p}) : \eta^{\mu p} = 0\}$ ,  $\Theta = 0$

Its equations of motion split naturally into rheotropic conditions [1]

$$E^a = e^a, \quad E^i = 0, \quad \Rightarrow \quad dX^{\underline{m}} = e^a u_{\underline{a}}^{\underline{m}} \quad (6)$$

and proper dynamical equation  $u^{\underline{im}} \delta S / \delta X^{\underline{m}} = 0$ , which can be written in terms of the pull-back  $\Omega^{ai} = d\xi^m \Omega_m^{ai}$  of the covariant Cartan form  $\Omega^{ai} = u_{\underline{m}}^{\underline{a}} du^{\underline{im}}$  (5)

$$\Omega^{ai}(\nabla_a) \equiv e_a^m \Omega_m^{ai} = 0, \quad (7)$$

Passing from Eqs. (6) to their selfconsistency (integrability) conditions ( $ddX = 0 = d(e^a u_{\underline{a}}^{\underline{m}})$ ) we can exclude the embedding functions  $X(\xi)$  as well as harmonic fields  $u(\xi)$  ( $v(\xi)$ ) from the consideration and get the equations

$$e_a \Omega^{ai} = 0, \quad T^a \equiv \mathcal{D}e^a \equiv de^a - e_b \Omega^{ba} = 0, \quad (8)$$

written in terms of intrinsic vielbeins  $e^a = d\xi^m e_m^a$  and Cartan forms (5) only. The later satisfy the Maurer-Cartan equations

$$d\Omega^{\underline{ab}} - \Omega_{\underline{c}}^{\underline{a}} \Omega^{\underline{cb}} = 0, \quad \begin{cases} \mathcal{D}\Omega^{ai} \equiv d\Omega^{ai} - \Omega_b^a \Omega^{bi} + \Omega^{aj} \Omega^{ji} = 0, \\ R^{ab}(d, d) = d\Omega^{ab} - \Omega_c^a \Omega^{cb} = \Omega^{ai} \Omega^{bi}, \\ R^{ij}(d, d) = d\Omega^{ij} + \Omega^{ij'} \Omega^{j'j} = -\Omega^{ai} \Omega_a^j, \end{cases} \quad (9)$$

The equations (7) – (9) describe the minimal embedding of the  $p$ -brane world volume into the flat target space–time and are referred as geometric approach equations [3,2,7].

**3. Some properties of doubly supersymmetric geometric approach.** Independent equations of motion for generalized action (1) are the rheotropic conditions

$$E^a = \Pi^{\underline{m}} u_{\underline{m}}^a = e^a, \quad E^i = \Pi^{\underline{m}} u_{\underline{m}}^i = 0, \quad D_{\alpha q} \Theta^\mu v_{\underline{\mu} \alpha \dot{q}} = 0 \quad (10)$$

and proper dynamical equation  $(\gamma^a)_\beta^\alpha D_a \Theta^\mu v_{\underline{\mu} \alpha \dot{q}} = 0$ . It is important that Eq. (10)  $\Pi^{\underline{m}} = dX^{\underline{m}} - id\Theta \Gamma^{\underline{m}} \Theta = e^a$  contains, in particular, Geometrodynamical condition [8]  $\Pi_{\alpha \dot{q}}^{\underline{m}} = D_{\alpha \dot{q}} X^{\underline{m}} - iD_{\alpha \dot{q}} \Theta \Gamma^{\underline{m}} \Theta = 0$ , which was a starting point for previous constructions of world sheet superfield formulations of superparticles and superstrings [8,2], as well as the twistor–like solution  $\Pi_a^{\underline{m}} = D_a X^{\underline{m}} - iD_a \Theta \Gamma^{\underline{m}} \Theta = u_a^{\underline{m}}$  of the Virasoro–like constraint  $\Pi_a^{\underline{m}} \Pi_b^{\underline{m}} = \eta_{ab}$ .

The geometric approach equations for super– $p$ -branes [2,1,9] can be obtained by considering the integrability conditions for Eqs. (10) and for the relations

$$E^{\alpha q} \equiv d\Theta^\mu v_{\underline{\mu}}^{\alpha q} = e^{\alpha q}, \quad E^{\alpha \dot{q}} \equiv d\Theta^\mu v_{\underline{\mu}}^{\alpha \dot{q}} = e^a \psi_{a \alpha \dot{q}}, \quad (11)$$

which collect the third equation from (10) and conventional rheotropic conditions [1,9]. They are formulated in terms of the world volume vielbein  $e^A = (e^a, e^{\alpha q})$  and independent components of the pull–backs of the Cartan forms (5), one of which coincides with the superfield  $\psi_a^{\alpha \dot{q}} = D_a \Theta^\mu v_{\underline{\mu}}^{\alpha \dot{q}}$  (11) [2,9].

It shall be stressed, that *the world volume supergravity constraints follows from the equations of motion for the generalized action (1)*. Indeed, the integrability conditions for the first of Eqs. (10) acquire the form  $T^a = De^a \equiv de^a - \Omega_b^a e^b = -2ie^{\alpha q} e^{\beta p} \gamma_{\alpha \beta}^a$  after some algebra. This equation contains the most essential supergravity constraint  $T_{\alpha p \beta q}^a = -2i\gamma_{\alpha \beta}^a \delta_{pq}$ . Other constraints of world volume supergravity are conventional and also appear as selfconsistency conditions, because we use the freedom of redefinition of the geometrical quantities, which are absent in the action, to fix them to be induced by the embedding.

**4.** In conclusion, let us consider bosonic  $p$ -brane interacting with generalized Kalb–Ramond field  $B_{p+1} = dX^{\underline{m}_{p+1}} \dots dX^{\underline{m}_1} B_{\underline{m}_1 \dots \underline{m}_{p+1}}(X)$ . The action functional  $S = S_{D,p}^0 + S_{D,p}^{int}$  is the sum of the free  $p$ -brane action [6] (see item 2) and the interaction term  $S_{D,p}^{int} = -\int_{\mathcal{M}^{p+1}} B_{p+1}$ . It can be proved [4], that the geometric approach equations (8)–(9) (which describes world volume as a surface embedded into space–time) remains the same for the system under consideration, and Eq. (7) is replaced by

$$\Omega^{ai}(\nabla_a) = e_a \Omega^{ai} = \frac{1}{(p+1)!} \epsilon_{a_0 \dots a_p} u^{a_0 \underline{m}_0} \dots u^{a_p \underline{m}_p} u^{i \underline{m}_{p+1}} H_{\underline{m}_0 \dots \underline{m}_{p+1}}(X(\xi)),$$

where  $H_{\underline{m}_0 \dots \underline{m}_{p+1}}(X(\xi)) = (p+1) \partial_{[\underline{m}_0} B_{\underline{m}_1 \dots \underline{m}_{p+1}]}(X(\xi))$ .

Hence, the world volume is embedded as a nonminimal surface and its main curvature is defined by the field strength of the generalized Kalb–Ramond field, *which can be considered as arbitrary function of the coordinates  $X(\xi)$* .

The general theorems about local isometric embedding [10] guarantees that, if the dimension of target space time is  $D \geq (p+1)(p+2)/2$ , then we can describe arbitrary curved  $d = (p+1)$ - dimensional surface in such a way.

Hence, the considered model can be regarded as a model for description of  $d = (p+1)$  - dimensional gravity providing the dynamical ground for the embedding approach used before for investigation of General Relativity [10]. This realizes the idea of Regge and Teitelboim [11] about string- like description of Gravity.

The model for  $d = 4$  supergravity is provided by  $D = 10$  3- brane <sup>\*</sup>.

The generalized action principle provides a ground for supersymmetric generalization of such construction.

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<sup>\*</sup>It is interesting that the number  $D = 10$  of space time dimensions is distinguished by superstring theory, and that a 3- brane supersymmetric soliton exists in  $D = 10$  type *IIB* superstring theory [12]. Moreover, this soliton is exceptional in some reasons [12].